Finite Quantum Dynamics

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Abstract

We general-quantize the dynamics of the quantum harmonic oscillator to obtain a covariant finite quantum dynamics in a finite quantum time. The usual central ("superselected") time results from a self-organization. Unitarity necessarily fails, imperceptibly for middle times and grossly near the beginning and end of time. Time and energy interconvert during space-time decondensation or meltdown, at a rate governed by a constant like the Planck power.

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1   Singular theories, singular results

A group must be semisimple to be stable against experimental error (regular, robust, generic) [Segal, Vilela]. We suppose that special relativity, general relativity and quantum theory are transitory phases in the evolution of physical groups from singular ones based on axioms to stable ones based on experiment, through variations on one underlying regularization process, *general quantization* — introducing small generic non-commutativities that convert a non-semisimple algebra into a simple one.

To unify quantum theory and general relativity some hold one of the two theories fixed and adapt the other; we have made such efforts ourselves. But there are deep errors in both theories that neither correct. Now we take a more general perspective that embraces both quantization and special relativization as limiting cases.

From this viewpoint the three main evolutions of physics in the early twentieth century have suggestive similarities.

1. Each introduced a small new non-commutativity.

2. The scale of each modification is set by a fundamental constant, so small that the predicted non-commutativity is indetectable in older, coarser experiments.

3. Each changed a singular algebra to a more generic one.

Thus boosts cease to commute in special relativity, parallel transports in general relativity, and filtrations in quantum theory. The respective fundamental constants are $1/c^2$, G, and $\hbar$. The algebras introduced are those of the Lorentz group, the Einstein group, and the unitary group.

The three evolutions seem to be instances of one general process that we call flexion, and when it increases the non-commutativity of dynamical variables, *general-quantization*. A general-quantization is a homotopy connecting from a singular algebra to one that is less singular, more generic, by introducing a growing non-commutativity, a generalized curvature. The concept if not the term is due to Segal [27] and is perhaps implicit in the work of Inönü and Wigner [21] and Flato [18], and is first made explicit and applied in the work of Vilela Mendes [32]. Segal suggested that further quantization would not only stabilize the theory but might improve its agreement with experiment and cannot hurt it. Varela began that process and we explore it here.

Our search for the quantum of time started from the broad idea that discreteness and continuity are reconciled within a quantum theory [14]. This was not an adequate guide, and it required problematic changes in the standard group structure that seemed apt to conflict with experiment. Segal's principle focuses the search for quantum time and transforms this problem into a solution. The quantum theory must have a semisimple group to be stable (generic), and any group is infinitesimally close to a semisimple one. Therefore a change in the commutation relations of an existing unstable theory is desirable, and a sufficiently small change suffices to stabilize the group, accord with past data, and make critical predictions for future data.

A semisimple group that provides stability also permits a finite-dimensional representation, with bounded as well as discrete spectra for all operators, including time. Full stabilization can be expected to eliminate all infinities. Theories that are singular in having non-generic groups are also singular in making infinite predictions. We need not distinguish between these usages of the term singularity. Singular groups are not only unstable, they make physics blow up.

To deal with anti-commutation relations as well as commutation relations, we generalize the simplicity principle from Lie algebra to graded Lie algebra:

> Graded commutator algebras of isolated physical systems are simple.

This leads back to Clifford-Wilczek statistics, a general-quantized fermionic statistics already introduced by Wilczek based on Clifford rather than Grassmann algebra [33].

General quantization usually introduces new regularization operators or *regulators* for some commutators that are 0 in the singular limit, and new regularization constants or *regulants* for the final values of homotopy parameters. Special relativization and canonical quantization are exceptional. They have regulants c and $\hbar$ and no regulators. For general relativity, the curvature tensor is a regulator and the gravitational constant G is its regulant.

All these theories preserved or introduced some singularities that must now be regularized. All the singularities of present physics can be traced to their source in various singular Lie algebras. L. H. Thomas, H. P. Snyder, W. de Sitter, and I. E. Segal performed famous quantizations that reduce these singularities but still do not eliminate them, not even when are all combined and applied at once. To make the present physical algebras semisimple takes a quantization engineered for that purpose.

While simplicity seems like a special case of semisimplicity, in quantum theory one measurement reduces the semisimple algebra to one simple subalgebra. So we shall require simplicity without losing generality.

The simplicity principle implies much of quantum theory. It excludes classical mechanics, since the canonical group is not a Lie group. The simple Lie group that replaces the canonical group is the connected isometry group of some quadratic space up to isomorphism, according to Cartan. We may use this space as the state-vector space of the system, and its quadratic form as the scalar product of the quantum theory.

The simplicity principle is suggested by the fact that the evolution of physical theories is at least partly Darwinian. Other things being equal — which is not always the case — stable theories are better adapted to survive small improvements in measurements. Finiteness is then a somewhat unexpected reward for stability. Present divergent theories are but singular limits of finite simple generic ones. We can construct candidate simple theories by quantizing the singular theories that work best.

Theories that are proposed as fundamental are usually selected to be special or distinguished as opposed to generic, for reasons of apparent simplicity. For example, Euclid might have preferred his plane geometry to spherical geometry on the grounds that a flat space is "simpler" than a curved one. String theory and gauge theory in their original forms, for example, are set in commutative spaces and so preserve one of the main instabilities of the present quantum field theory. Gauge groups have arbitrary functions on space-time as group parameters and are not Lie groups. They must be general-quantized to make them Lie.

Deformation quantization [18] uses a homotopy too but not the simplicity principle. The canonical quantum oscillator, with all its singularities, seems to be an acceptable end point for deformation quantization, while it is only the starting point of Segal's quantization. The theory resulting from deformation quantization seems to have both a singular algebra product and a more generic one while nature and the general-quantized theory seem to have but one product.

The canonical Lie algebra $d\mathrm{H}_1(x, \partial_x, 1)$ of the differential calculus, quantum theory and gauge theory is unstable, singular, non-generic. Therefore it is probably just a transient phase that we will outgrow. Segal stabilized it by supplementing $\hbar$ with two new quantum constants $\hbar'$, $\hbar''$ of other dimensions. This produced a simple group that has the canonical algebra as a singular limiting case, has continuous symmetries that are nearly the same in a limited experimental domain and that can nevertheless become as large as we like, and is nevertheless finite in volume and dimension; as the round Earth is practically flat in a limited domain, yet finite. One might say that we have just fully grasped how round the world really is.

Unlike lattice regularization, general quantization does not reduce the relativity group of the space-time to a discrete subgroup. It merely changes the group slightly, ultimately to a simple Lie

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group. Snyder's quantum space-time and de Sitter's curved space-time were limited forms of general quantization.

Looking at physics past, Inönü and Wigner used a special case of Segal's general concept to relate special relativity to Galileo relativity. They formulated the concept of group contraction [21], a direct process, as a linear transformation of the Lie algebra that flattens it in the limit. Contraction destabilizes the theory. The Snyder and de Sitter general-quantizations were inverse contractions. The quantization of quantum theory suggested by Segal is not an inverse contraction, being necessarily non-linear.

The simplicity principle was in general circulation by the 1960's. Peter Bergmann mentioned it in a lecture, for example. And some form of Segal's quantization of the time-independent harmonic oscillator is now under study by several groups from several points of view [1, 8, 23, 24, 31]. But the first work to take theory-stability seriously is that of Vilela Mendes.

Looking to physics future, such reforms, pushed to their limit, lead not only to a stable theory but also to a finite one in a finite quantum space-time. These have been sought by some physicists since the formulation of quantum theory.

To stabilize present-day instabilities requires several radical changes at once. Present theories are *pointed* — have absolute space-time points — and *local* — couple these points only to their infinitesimal neighbors. The Einstein group (of a manifold) —like any other gauge group — is not simple precisely because it respects points, coupling x^μ into ∂_μ but not conversely. Such non-reciprocity is an infallible sign of a compound group. When we make the Einstein group simple and generic, we lose the space-time points. A simple physics that reduces to a gauge theory in a singular limit must be not merely non-local but non-pointed.

More generally, any local theory is singular and unstable, but an arbitrarily small quantization suffices to make it simple and stable. It is then non-local and non-pointed.

Planck introduced the constant $\hbar$ in a way that froze out the very stiff oscillators responsible for the infinite heat capacity of cavity radiation in Maxwell's theory. Einstein recognized this effect as a consequence of the quantum of radiation, the photon. The canonical commutation relations and complementarity provided the conceptual framework for this otherwise mysterious regularization. They left the zero-point energy of the resulting quantum theory of electromagnetism still divergent. To be sure, Heisenberg later replaced the local Lagrangian and Hamiltonian of Maxwell by a non-local reordering that was arbitrarily tailored to have zero ground-state energy-density. This zero is not a prediction of the theory but an arbitrary assumption. One expects any field theory to contribute to the zero-point energy of the vacuum, and so to a dark energy and mass, but present singular theories like quantum electrodynamics cannot predict this contribution.

The quantum theory of the linear harmonic oscillator, a constituent of all present quantum field theories, carries the seeds of some of the divergences of quantum field theory. Almost all the operators of this theory are unbounded, including its fundamental observables q, p. The basic operators of position q, momentum p, and Hamiltonian $H \sim \frac{1}{2}(p^2 + q^2)$ diverge on almost every vector ψ in its Hilbert space: $q\psi = p\psi = H\psi = \infty$. Since Von Neumann taught how to extract finite answers from such a mathematical mine-field we have become inured to life on the brink of infinity. But such infinities do not result from experiment but from cosmological assumptions that go beyond experiment to ideology. They occur in a quantum theory if and only if its Hilbert space is infinite-dimensional. Finite-dimensional Hilbert spaces cannot represent singular algebras well; but they can represent simple ones, and so simple or generic groups makes a finite theory possible with no loss of continuous symmetry.

By the *kinematical group* of a system we mean the group of all possible reversible actions on the system. We replace many of the present principles of quantum theory by the Segal simplicity

condition, which excludes classical mechanics and classical field theories:

The kinematical group of an isolated physical system is simple.

By a finite or simple quantum theory we mean one obeying this simplicity condition. Each simple Lie algebra is the Lie algebra of the isometry group of a unique finite-dimensional quadratic space; this is the state-vector space of the system. In a simple quantum theory all observables have finite bounded spectra. Infinite-dimensional algebras can still be entertained, namely as singular limits, for mathematical convenience; just as the differential calculus is sometimes a convenient approximation to the calculus of finite differences. But the symmetries of nature should already appear in the simple theory, which is the more physical theory, and not only in its singular limit, which is less physical.

General quantization does not make a theory that satisfies us aesthetically. We are all habitual transgressors of Ockham's Law. ("Thou shalt not multiply entities unnecessarily.") If three data bars will admit a straight line, we prefer the infinite line to the many large circles that also pass through the data bars, though some circle almost certainly fits the data better, and is shorter. This multiplies points unnecessarily. Likewise, from our finite experience with many space-time points we postulate that the set of space-time points is infinite both in the large and in the small, again breaking Ockham's Law. Segal's reform reaffirms Ockham's Law. Somehow a large numerical constant offends our sensibility when ∞ does not.

We test the simplicity principle here on the stationary (time-independent) and the dynamical (time-dependent) linear harmonic oscillator.

General-quantizing the stationary oscillator introduces two post-quantum Segal constants $\hbar', \hbar''$ besides the usual Planck constant $\hbar$. The regularized coordinate and momentum now transform as infinitesimal rotations in SO(3). We study how the resulting finite quantum oscillator approaches the usual singular quantum oscillator as a singular limit.

General quantization freezes out the offending zero-point oscillations of extremely hard or soft oscillators without greatly changing the zero-point energies of medium ones. The system Hilbert space becomes finite-dimensional and its operations can in principle always be carried out. The frozen oscillators also grossly violate the usual equipartition and uncertainty relations. Algebra regularization clearly has profound consequences for extreme energy physics: the physics of both very high and very low energies.

This toy model illustrates how a finite quantum theory of the cavity can produce a finite zero-point energy without conflicting with the many finite predictions and symmetries of the usual quantum theory. We propose that the linear harmonic field oscillators considered fundamental in present quantum physics – we mean those of allegedly fundamental fields, not those of crystals, say — are actually dipole rotators in a three-dimensional space, with fixed high angular-momentum quantum number l and with third angular-momentum component $m \sim l$. The unobserved oscillators responsible for the infrared and ultraviolet divergencies of present quantum theories are frozen by finite quantum effects described here and contribute negligibly to the zero-point energy.

We then general-quantize the quantum oscillator dynamics. This leads to a finite quantum theory of the dynamical harmonic oscillator in quantum time.

In dynamics the oscillator or field variable is a function of time. When we call a theory c/c, q/c, or q/q, the denominator tells whether the independent temporal or T variable is c or q, and the numerator tells about the dependent or S variable. Quantum chromodynamics is a q/c theory. Its field-variable system is quantum, with a non-commutative logic, and the space-time is c, with a commutative logic. The commutative logic leads to a singular algebra and requires flexing. Here we quantize the quantum oscillator q/c dynamics and arrive at a q/q one that is stable and finite.

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2 Regularization by quantization

The totality of Lie products $\mathsf{x} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ admitted by a given vector space $\mathcal{A}$, also called structure tensors, form a quadratic sub-manifold $\mathcal{J}(\mathcal{A}) = \{\mathsf{x}\}$ of the tensor space $\mathcal{A} \otimes [\mathcal{A}^\dagger \otimes \mathcal{A}^\dagger]$ (here the $\dagger$ dualizes and the brackets skew-symmetrize), defined by the Jacobi law, which is quadratic in x. A *regular* (stable, robust) algebra is a Lie algebra that is unchanged up to isomorphism by all sufficiently small changes in its structure tensor (Lie product) within the manifold $\{\mathsf{x}\}$. For example, the Lorentz algebra is stable against small corrections to the speed of light. By *flexion* we mean a homotopy of the structure tensor of a compound algebra that it less commutative, closer to semisimple. When this makes the dynamical variables non-commutative it becomes quantization. *Flattening* is the inverse process.

2.1 Flexing the canonical commutation relations

The canonical (or Heisenberg) Lie algebra $d\mathrm{H}(1)$ is defined by the canonical commutation relations

$$p \mathsf{x} q = -i\hbar 1, \quad 1 \mathsf{x} q = 0, \quad q \mathsf{x} 1 = 0. \tag{1}$$

among its three hermitian generators $q, p, 1$. It is compound and the central unit 1 generates its radical. 1 is an idol of the theory in the sense of Bacon [2]. Segal proposed to simplify $d\mathrm{H}(1)$ by introducing a third variable r to replace 1, and two more quantum scale constants, which we designate here by $\hbar' \equiv \hbar^{[1]}$ and $\hbar'' \equiv \hbar^{[2]}$. We also switch from hermitian observables $p, q, 1$ to anti-hermitian generators $\widehat{p}, \widehat{q}, \widehat{r}$. Segal's general-quantized commutation relations are, except for notation,

$$\widehat{q} \mathsf{x} \widehat{p} = \hbar\widehat{r}, \quad \widehat{r} \mathsf{x} \widehat{q} = \hbar'\widehat{p}, \quad \widehat{p} \mathsf{x} \widehat{r} = -\hbar''\widehat{q}, \tag{2}$$

[8, 32, 27] For any $\hbar$, $\hbar'$, $\hbar'' > 0$ these relations define the Lie algebra $d\mathrm{SO}(2,1)$. The irreducible unitary representations of this non-compact group are infinite-dimensional. Ultimately we will need an indefinite metric for relativistic reasons, but not for the time-independent harmonic oscillator. We therefore drop the minus sign and adopt the general-quantization

$$\widehat{q} \mathsf{x} \widehat{p} = \hbar\widehat{r}, \quad \widehat{r} \mathsf{x} \widehat{q} = \hbar'\widehat{p}, \quad \widehat{p} \mathsf{x} \widehat{r} = \hbar''\widehat{q}, \tag{3}$$

with constants $\hbar, \hbar', \hbar'' > 0$ and group $\mathrm{SO}(3)$.

Let us rewrite $q \equiv q^{[0]}$, $p \equiv q' \equiv q^{[1]}$, $o \equiv q'' \equiv q^{[2]}$ and assume an invariant Euclidean metric g_{ij}. Then [4, 19]

$$\widehat{q}^{[i]} \mathsf{x} \widehat{q}^{[j]} = -i \sum_k \epsilon^{ij}{}_k \hbar^{[k]} \widehat{q}^{[k]}. \tag{4}$$

We call $\widehat{p}$, $\widehat{q}$, $\widehat{r}$ momentum, position, and action generators, respectively. In this toy, $\widehat{r}$ is the sole regulator.

This general-quantized algebra looks as if one has replaced the imaginary quantum constant $\hbar i$ by a dynamical variable $\widehat{r}$, a process one might call "i activation." The relations among $\widehat{p}, \widehat{q}, \widehat{r}$ are symmetric under $\mathrm{SO}(3, \mathbb{R})$. Where the canonical theory has an absolute relation of canonical conjugacy expressed by $[p, q] = -i\hbar$, the general-quantized theory has a relative relation of *canonical conjugacy with respect to r*, expressed by $[\widehat{p}, \widehat{q}] = \hbar\widehat{r}$. For example, the canonical conjugate of $\widehat{q}$ with respect to $\widehat{p}$ is $-\widehat{r}$.

In a previous exploration in quaternion quantum theory an activated $\hbar i$ served as Higgs field defining the electromagnetic axis $\eta(x)$ that resolves the electroweak gauge boson into electromagnetic and weak bosons [12], and gives mass to the charged partner of the photon through the Stückelberg-Higgs effect. This led to a natural SU(2) that was interpreted as isospin. The activated $i\hbar$ generated

rotations about the electric (or electromagnetic) axis in isospin space, defining a natural Higgs field. The quaternion theory was dropped because it did not shed light on color SU(3).

Now we activate $\hbar i$ on the more principled grounds of Segal: simplicity. We expect that the third axis will again give rise to a Higgs field. There is now plenty of room for internal groups like color SU(3), though we do not seek them for the harmonic oscillator.

Flexing generally faces the same kind of factor-ordering problems as quantization and gauging.

Except for scale factors the simplified commutation relations are those of an SO(3) quantum angular-momentum operator-valued vector $\mathbf{L} = i\mathbf{L} \times \mathbf{L}$ for a dipole rotator in three-dimensional space. We assume an irreducible representation with

$$\mathbf{L}^2 = l(l+1) \tag{5}$$

where l can have any non-negative half-integer eigenvalue. (In the present work it suffices to consider only integer values of l.) Then L_1, L_2, L_3 are represented by $(2l+1) \times (2l+1)$ matrices obeying

$$L^i \times L^j = -i\epsilon^{ijk} L_k. \tag{6}$$

We fix the scale factors by setting

$$q^{[k]} = \mathsf{Q}^{[k]} L^{[k]} \quad \text{(No sum over } k.) \tag{7}$$

In the singular limit $l \to \infty$ and the oscillator is nearly polarized along the L_3 axis, with $L_3 \approx l$. By (6)

$$\mathsf{Q} = \sqrt{\hbar \hbar'}, \quad \mathsf{Q}' = \sqrt{\hbar \hbar''}, \quad \mathsf{Q}'' = \sqrt{\hbar' \hbar''} = 1/l. \tag{8}$$

The commutation relations (6) and the angular momentum quantum number l determine a simple (associative) enveloping algebra $\mathrm{A}(\mathbf{L}, l)$ of $(2l+1) \times (2l+1)$ matrices. The spectral spacing of L_3 is 1, so the finite quantum constants $\mathsf{Q}, \mathsf{Q}', \mathsf{Q}''$ serve as quanta of the position, momentum and action variables. Since q, p have continuous spectra in quantum theory, the constants Q, Q' must be very small on the ordinary quantum scale. It follows that $\mathsf{Q}'' = \mathsf{Q}\mathsf{Q}'/\hbar$ is also very small on that scale and $l \gg 1$.

For $l \gg \sqrt{l} \gg 1$, variations $\delta(\hat{r}^2) \leq O(l^{-1/2}) \ll 1$ about $\hat{r}^2 = -1$ can be negligible at the same time as the spectral intervals $\delta p \leq \mathsf{Q}' \sqrt{l}$ and $\delta q \leq \mathsf{Q} \sqrt{l}$ for quasicontinuous $p, q \approx 0$. This simulates the usual oscillator kinematics.

3 Stationary harmonic oscillator

In the section we general-quantize and regularize the time-independent linear harmonic oscillator (§3), recapitulating and extending the work of [29], and then the time-dependent one (§4), with additional differential operators t and ∂_t.

General-quantizing the oscillator modifies the statistics. The oscillator is the prototype bosonic aggregate. Its Heisenberg algebra is that of the creators and annihilators of structureless bosons. When we change that algebra from that of an oscillator to that of a rotator, we change the statistics from bosonic to one that may be called finite-bosonic. There is no bound on the boson occupation number, but the finite boson has a finite bound N on its occupation number. N is one of the regulants of the regularized theory. When $N \to \infty$, the finite-bosonic statistics approaches the bosonic statistics of the singular theory.

A familiar ordering question arises at once. Since general-quantizing introduces non-commutativity, the factor-ordering of some products is immaterial in the flat theory but is significant in the general-quantized theory. Re-ordering the singular theory will introduce a correction of order $h'h''$ that may

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be experimentally significant. This makes general-quantization as notation-dependent and ambiguous as quantization and relativization.

The Hamiltonian of the general-quantized harmonic oscillator is

$$H = \frac{\mathsf{Q}'^2}{2m} L_2^2 + \frac{k\mathsf{Q}^2}{2} L_1^2 =: \frac{K}{2} \left(L_2{}^2 + \kappa^2 L_1{}^2 \right) \tag{9}$$

where

$$K := \frac{(\mathsf{Q}')^2}{m}, \quad \kappa^2 = \frac{\hbar' m k}{\hbar''}. \tag{10}$$

For fixed $\hbar^{[k]}$, all finite oscillators are divided into three kinds with ill defined boundaries: *medium*, where kinetic and potential terms in H are of comparable size ($\kappa \sim 1$); *soft*, when the potential energy term is dominant ($\kappa \to 0$); and hard, when the kinetic energy term is dominant ($\kappa \to \infty$). For a scalar field in standard singular quantum field theory, the oscillators that give rise to infrared divergencies become soft oscillators in the finite quantum theory, and those that feed ultraviolet divergencies become hard oscillators.

To represent the general-quantized time-independent algebra, let $\gamma_\sigma(n)$ ($\sigma = 1, 2, 3$, $n = 1, \ldots, N$) be $3N$ Clifford generators of positive signature. Each $\gamma_\sigma(n)$ represents an elementary process that toggles the occupation number of one "chronon" — quantum of space-time-etc. — with Clifford-Wilczek statistics.

Then we take

$$
\begin{aligned}
q &= \mathsf{Q} \sum_n \gamma_{31}(n), \\
p &= \mathsf{P} \sum_n \gamma_{23}(n), \\
r &= \mathsf{R} \sum_n \gamma_{12}(n)
\end{aligned}
\tag{11}
$$

as regularized anti-coordinate, anti-momentum, and anti-action of the oscillator at one instant of time.

Assume that N is even. Then the spinors on which these operators act have $2^{3N/2}$ components before reduction of the matrix algebra into irreducible representations.

3.1   Medium oscillators

The case $\kappa = 1$ is symmetric under rotations about the z axis, and so is especially simple [31]. Since

$$(L_1)^2 + (L_2)^2 + (L_3)^2 = \mathbf{L}^2 = l(l+1), \tag{12}$$

$$H = \frac{K}{2} \left(l(l+1) - (L_3)^2 \right) \tag{13}$$

The oscillator quantum number n that labels the energy level is now

$$n = l + m. \tag{14}$$

The general-quantized energy spectrum is

$$E_n = \frac{K}{2} \left(l(l+1) - (n-l)^2 \right) = lK \left(n + \frac{1}{2} - \frac{n^2}{2l} \right) \tag{15}$$

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For $n \ll \sqrt{l} \ll l$ this reproduces the usual uniformly-spaced oscillator energy spectrum as closely as desired, but with multiplicity 2 for each level instead of 1.

The ground-state energy for this oscillator is

$$E_0 = \frac{1}{2}Kl = \frac{1}{2}\hbar\omega, \tag{16}$$

exactly the usual oscillator ground energy, since $Kl = \hbar\omega$.

The main new feature is that this finite oscillator has an upper energy limit

$$E_{max} = \frac{1}{2}Kl(l+1) \tag{17}$$

as required by a finite quantum theory.

In the general case of $\kappa \sim 1$ we obtain an upper bound for the ground energy by a variational approximation with the trial function $|L_3 = \pm l\rangle$. This reproduces our previous result (16), now as an upper bound for the ground energy of a medium FLHO:

$$E_0 \leq \frac{1}{2}Kl. \tag{18}$$

Medium oscillators have many states with m-value close to the extremum values $m = \pm l$. The usual Heisenberg uncertainty principle

$$(\Delta p)^2(\Delta q)^2 \geqslant \frac{1}{4}\langle ip \times q\rangle^2 = \frac{\hbar^2}{4}. \tag{19}$$

becomes

$$(\Delta L_1)^2(\Delta L_2)^2 \geqslant \frac{\hbar^2}{4}\langle L_3\rangle^2_{|L_3 \approx \pm l\rangle} \tag{20}$$

for a low-lying energy level of a medium oscillator. By (7) and (8),

$$(\Delta p)^2(\Delta y)^2 \geqslant \frac{\hbar^2}{4} \tag{21}$$

for large l. So medium oscillator states in low-lying energy levels have uncertainty products consistent with the Heisenberg uncertainty principle.

3.2 Soft oscillators

When $\kappa \ll 1$ we can estimate the spectrum of H by perturbation theory. The unperturbed Hamiltonian is the kinetic energy

$$H_0 = \frac{K}{2}L_1^2. \tag{22}$$

The unperturbed eigenvectors are $|L_1 = m\rangle$. The unperturbed energy levels are

$$E_m(0) = \frac{K}{2}m^2. \tag{23}$$

The first-order shifts are

$$\delta E_m = \frac{K}{2}\langle L_1 = m|L_2^2|L_1 = m\rangle. \tag{24}$$

Due to the axial symmetry of $|L_1 = m\rangle$,

$$\langle L_1 = m|L_2^2|L_1 = m\rangle = \langle L_1 = m|L_3^2|L_1 = m\rangle. \tag{25}$$

Therefore the energy shift is

$$\frac{K}{2}\langle L_1 = m|\kappa^2 L_2^2|L_1 = m\rangle = \frac{K}{4}\kappa^2\langle m|L_1^2 + L_2^2|m\rangle$$
$$= \frac{K}{4}\kappa^2\langle m|L^2 - L_3^2|m\rangle$$
$$= \frac{K}{4}\kappa^2 l(l+1) - m^2 \tag{26}$$

to lowest order in κ^2, The general-quantized energy spectrum is then

$$E_m \approx \frac{K}{2}m^2 + \Delta E_m$$
$$= \frac{K}{2}m^2 + \frac{1}{4}K\kappa^2\left[l(l+1) - m^2\right] \tag{27}$$

The estimated upper bound for the energy is

$$E_{max} \approx \frac{1}{2}Kl^2(1 + \frac{\kappa^2}{2l}) \tag{28}$$

For $\kappa \to 0$ this reproduces the upper bound for the unperturbed Hamiltonian L_3^2, as it should. The zero-point energy E_0 of first-order perturbation theory is

$$E_0 \approx \frac{1}{4}\kappa^2 K l(l+1) \tag{29}$$

For $\kappa \to 0$ this is infinitesimal compared to the standard quantum oscillator.

A soft oscillator shows little resemblance to the usual quantum oscillator. Its energy levels do not have uniform spacing. Its kinetic energy dwarfs its potential energy, grossly violating equipartition. The low energy states are near $|L_1 = 0\rangle$ instead of $|L_3 = \pm l\rangle$. Its p degree of freedom is frozen out. It is "too soft to oscillate:" There is not enough energy in the q degree of freedom, even at its maximum excitation, to produce one quantum of p. The uncertainty relation reads

$$(\Delta L_1)^2(\Delta L_2)^2 \geqslant \frac{\hbar^2}{4}\langle L_3\rangle^2_{|L_1\approx 0\rangle} \approx 0 \tag{30}$$

Therefore

$$\Delta p\Delta q \ll \frac{\hbar}{2}, \tag{31}$$

which violates the Heisenberg uncertainty principle grossly.

3.3 Hard oscillators

Hard oscillators reverse the story but violate the same basic principles of the caninical quantum theory as soft oscillators. A hard oscillator has much greater potential than kinetic energy. Its low energy states are now near $|L_2 = 0\rangle$ instead of $|L_3 = \pm l\rangle$ (the medium case) or $|L_1 = 0\rangle$ (the soft case). Its q degree of freedom is frozen out. It is "too hard to oscillate." There is not enough energy in the p degree of freedom, even at maximum excitation, to arouse one quantum of q.

A hard oscillator can likewise be treated by perturbation methods. The kinetic energy is the perturbation. We may carry all the of the main results in the previous section for soft oscillators to

the hard ones simply by replacing κ with $1/\kappa$ and K with $K\kappa^2$. A hard oscillator shows no resemblance to the usual quantum oscillator. Its zero-point energy E_0 is now

$$E_0 \approx \frac{K}{4}l(l+1) \tag{32}$$

For $\kappa \to \infty$ this is infinitesimal compared to the usual quantum oscillator zero-point energy. Its energy levels of a hard oscillator are not uniformly spaced. Its uncertainty relation reads

$$(\Delta L_1)^2 (\Delta L_2)^2 \geqslant \frac{\hbar^2}{4} \langle L_3 \rangle^2_{|L_2 \approx 0} \approx 0 \tag{33}$$

Therefore

$$\Delta p \Delta q \ll \frac{\hbar}{2}, \tag{34}$$

which seriously violates the Heisenberg uncertainty principle again.

3.4 Unitary Representations

Variables p and q do not have finite-dimensional unitary representations in classical and quantum physics. They are continuous variables and generate unbounded translations of each other. But since in the general-quantized quantum theory all operators become finite and quantized, we expect all translations to become rotations, with simple finite-dimensional unitary representations.

The canonical group of a classical oscillator becomes the unitary group of an infinite-dimensional Hilbert space for a quantum oscillator, and the unitary group of a $2l+1$ dimensional Hilbert space for the general-quantized quantum oscillator.

The Lie algebra generated by momentum and position as infinitesimal symmetry generators is $\mathbf{H}(1)$ for the classical and quantum oscillator and the $\mathbf{SO}(3)$ angular momentum algebra for the finite oscillator. The corresponding Lie algebras are the Heisenberg Lie algebra $d\mathrm{H}(1)$ and the orthogonal-group Lie algebra $d\mathrm{SO}(3)$.

The commutation relations $\mathbf{L} \times \mathbf{L} = -i\mathbf{L}$ and the angular momentum quantum number l determine a simple matrix algebra $\mathrm{A}(\widehat{\mathbf{L}}, l)$; here l can be any non-negative half-integer. The spectral spacing of the operators L_k is 1, so the constants $\hbar_k$ serve as quanta of the $\widehat{q}_k$ respectively. Since q, p have continuous spectra in the flattened quantum theory, their quanta $\hbar_1, \hbar_2$ must be small on the ordinary quantum scale $\hbar \sim 1$. It follows that $\hbar_3$ is also small on that scale.

For $\sqrt{l} \gg 1$, variations $\delta(\widehat{r}^2) \leq O(l^{-1/2}) \ll 1$ from $\widehat{r}^2 = 1$ can be negligible at the same time as the spectral intervals $\delta p \leq \hbar_2 \sqrt{l}$ and $\delta q \leq \hbar_1 \sqrt{l}$ for quasicontinuous $p \ll \hbar_2, q \ll \hbar_1$.

In the canonical quantum theory, q-translation is a continuous one-parameter unitary subgroup of the kinematical group, but not in the simple quantum theory, obviously. If the infinitesimal advance of q were represented by a hermition operator, q would have a continuous spectrum, and the Hilbert space would be infinite dimensional, and all values of q would have the same spectral multiplicity. In fact the Hilbert space is finite-dimensional and each eigenvalue $\lambda = iq'$ of iq has a multiplicity $M(\lambda)$ that is maximum for $q^2 = 0$, varies slowly with q^2 for $q \sim 0$, and goes to 0 linearly in λ.

3.5 Quantum internal space

It is natural to ask what space the oscillator moves in. In c physics we usually describe a system by a set of states, or state-set, with some structure.

In quantum theory we may specify a simple system $\mathcal{T}$ by an associated finite-dimensional state-vector space, that we designate by $\mathrm{V}\mathcal{T}$, or by its algebra of coordinates $\mathrm{A}\mathcal{T} = \mathrm{Endo}\,\mathrm{V}\mathcal{T}$. The

kinematics of the q oscillator at one time is defined by an infinite-dimensional irreducible representation of the three-dimensional complex Heisenberg algebra $d\mathrm{H}(1) = \mathrm{A}[q, p; \mathbb{C}]$.

The general-quantized theory replaces $\hbar i \in \mathbb{C}$ by r $\hbar i \widehat{r}$, with regulator r. We naturally define the corresponding quantum phase space by the complex algebra $\mathrm{A}[\widehat{q}, \widehat{p}, \widehat{r}; \mathbb{C}] = d\mathrm{SO}(3)$ with three generators $q \sim L_1$, $p \sim L_2$, $r \sim L_3$ isomorphic to the three components of a three-dimensional angular momentum vector $\mathbf{L} = (L_1, L_2, L_3)$. This small change in the commutation relations changes the spaces drastically in the large. The three variables L_1, L_2, L_3 are on the same footing in the regular theory and are related to a central quantum number l by $(L_1)^2 + (L_2)^2 + (L_3)^2 = l(l+1)$. Now the irreducible representations are finite-dimensional, of dimension $2l+1$.

The regularized oscillator Hamiltonian $H = (p^2/2m) + (kq^2/2)$ is that of a rigid dipole rotator with one infinite principle moment of inertia $I_3 = \infty$ and with a sharp total angular momentum quantum number $l < \infty$. It has a finite number of states $\mathsf{N} = 2l + 1 \sim 1/(\hbar'\hbar'')$. In the sinular limit $\mathsf{N} \to \infty$, so N is a regulant. Manfredi and Salasnich discuss the statistical distribution of the levels of the energy spectrum of the triaxial rotator and point out that part of the rotator spectrum can approximate the spectrum of a linear harmonic oscillator [24].

Obviously the operator $\hbar\omega(L_1 2 + l))$ has exactly the energy spectrum of the singular theory, cut off at the Nth level. There is presumably a modification f $\widehat{H}'$ of $\widehat{H}$ that has exactly this equally spaced spectrum and differs from our quadratic $\widehat{H}$ by corrections that vanish in the singular limit. This equally spaced energy spectrum eliminates the interactions between the quanta of excitation when their number is less than N, while the impossibility of higher occupation than N can be regarded as an effective infinite repulsive $\mathsf{N} + 1$-body potential.

3.6   Comparison of regular and singular quantum oscillators

Let us compare the classical, quantum, and finite linear harmonic oscillators [28].

The general-quantized quantum oscillator is isomorphic to a dipole rotator with Hamiltonian of the special form

$$H = \frac{1}{2}K_x(L_x)^2 + K_y\frac{1}{2}(L_y)^2, \quad K_x = \frac{\mathsf{P}^2}{\mu}, \quad K_y = \frac{\mathsf{Q}^2}{\lambda}. \tag{35}$$

The classical and singular quantum oscillators have continuous coordinates and momenta. The finite-oscillator position and momentum variables are quantized with finite, uniformly spaced, spectra, with spacing P, Q respectively, and maximum values $l\mathsf{P}, l\mathsf{Q}$. To pass for the more familiar singular ungeneral-quantized oscillators of present-day physics, a general-quantized oscillator must have many states, $\mathsf{N} = 1/\mathsf{J} \gg 1$. But such oscillators are accompanied by some that have few states.

In the classical theory all oscillators are isomorphic up to scale. All singular quantum linear harmonic oscillators are likewise isomorphic up to scale. The constants $\hbar, \hbar''$ finally break this scale invariance. The finite quantum linear harmonic oscillators fall into three broad classes, which we term *soft*, *medium*, and *hard*, according to the dimensionless ratio $K_y/K_x = \kappa^2$ of maximum possible potential energy to maximum possible kinetic energy.

Medium oscillators ($\kappa \sim 1$) have $\sim \sqrt{N}$ low-lying states with nearly the same zero-point energy and level spacing as the QLHO, like rotators nearly polarized along the z axis with $L_z \sim \pm l$. They resemble the singular QLHO, obeying the Heisenberg uncertainty principle and the equipartition principle when they are in their low-lying energy levels.

The soft and hard FLHO's do not resemble the QLHO at all. Their low-lying energy states correspond to rotators with $\kappa \sim 0$ or $\kappa \sim \infty$. Their 0-point energy is infinitesimal compared to the QLHO. They grossly violate both the uncertainty principle and equipartition in all their states.

Soft oscillators have frozen momentum $p \sim 0$, their maximum potential energy being too small for even one quantum of momentum.

Hard oscillators (kinetic $\ll$ potential) have frozen position $q \sim 0$, their maximum kinetic energy being too small for even one quantum of position. These quantum freezings of degrees of freedom resemble but extend the original ones by which Planck obtained a finite thermal distribution of cavity radiation. Even the 0-point energy of a similarly regularized field theory will be finite, and can therefore be physical.

For the standard linear harmonic oscillator,

$$\widehat{E} - \widehat{H} = AL_{14} - BL_{23}^2 - CL_{24}^2 \tag{36}$$

with real positive constants A, B, C. For medium oscillators this approaches the singular theory as $\hbar_{rp}, \hbar_{rq} \to 0$. For soft oscillators the mass dominates the spring, $C \ll B$ and may be treated as a perturbation. The kinetic energy perturbation CL_{24}^2 happens to commute with the unperturbed energies $AL_{14} - BL_{23}^2$.

4 Dynamical harmonic oscillator

Now let us flex the dynamical or time-dependent oscillator. staying as close to the singular theory as regularity permits. Above all we maintain and extend the correspondence principle. The variables and equations of the general-quantized theory converge (non-uniformly) to those of the singular theory as $\mathsf{T} \to 0$ and $\mathsf{N} \to \infty$.

This is a critical test of the general-quantization strategy. We had not succeeded in making a reasonable q/q dynamical theory before now.

4.1 Forms of dynamics

Let us designate the q system under study at some one instant by $\mathcal{S}$, and the system of c times over which we study it by $\mathcal{T}$. Here we general-quantize a q/c dynamical theory of $\mathcal{S}$ over $\mathcal{T}$ into a q/q theory.

The standard Hilbert-space structure is not enough to formulate a dynamical theory; it must be supplemented by a theory of time. The canonical dynamics takes the time axis to be $\mathbf{R}$, postulates a fixed Hamiltonian $H(t)$ possibly depending on t, and assumes a dynamical equation of the form

$$i\hbar \frac{dq(t)}{dt} - H(t) \times q(t) = [E - H] \times q(t) = 0. \tag{37}$$

This dynamics does not relate mere observables like q but entities like $q(t)$ of a separate category, with separate meaning and structure. We begin by giving a commutator algebra for this larger structure.

The canonical algebra has a complete set of commuting variables all associated with one time, and the elements of the algebra are functions of only one time variable. This is a single-time or *synchronic* theory. Variables of a synchronic algebra are independent and grade-commute if they are associated with spatially separated events. Later variables, however, are not independent of early variables, but are identified with combinations of them determined by integrating the canonical equations of motion. This makes a non-relativistic distinction between space and time.

In the many-time or *diachronic* form of dynamical algebra there are independent grade-commuting variables at each space-time point, regardless of whether the separation is spacelike or timelike. The elements of the diachronic algebra represent functions of space-time points. The vectors on which

they act describe histories; we call them path vectors or history vectors. The dynamical equations are now subsidiary conditions, not operator equations. They single out a subspace of dynamically allowed path-vectors Ψ that satisfy dynamical equations of the vector form $L\Psi = 0$, instead of equations of the operator form $L = 0$ of the synchronic theory. where L is some linear operator to be specified.

One can determine a synchronic state-vector by measurements all at one time. To determine a diachronic path-vector one must suspend the canonical equations and make measurements at every time. Since we attribute the canonical equations to a condensation, this turning-off is not a purely mathematical fantasy. It may happen in a change of phase of the ether.

A fixed Hamiltonian is built into structure of the synchronic algebra. The diachronic algebra is completely defined without reference to any Hamiltonian.

These considerations apply to both the c/c and the q/c dynamics.

In this first study we general-quantize only the synchronic dynamics. We suppose that a definite Hamiltonian is valid from input to output, and use it to identify later variables with combinations of earlier ones. Any measurement overrides this Hamiltonian, so this identification runs only between input and output times.

The q/c synchronic dynamics gives time two special roles:

- t is a central operator, a superselection rule [26].

- Experiment presents us with dynamics as a system of unitary transformations $W(t, t')$ that connect any two values of time t, t' and the associated state-vectors $\psi(t), \psi(t')$.

Requiring all observables to commute with t is not generic. When t is non-central, the dynamical correlation between different times appears as an off-diagonal long-range order, off diagonal in t. Condensations create the central q's and p's of classical mechanics. It is convenient to use the same language for central time. We treat central time as if it resulted from a condensation of a more generic dynamics.

When we combine systems, quantities may combine in three useful ways:

1. Multiplicatively, like finite symmetry group operations;

2. Additively, like infinitesimal symmetries;

3. Identically, like time t and i in q/c theory [15].

We attribute the identical mode of composition to a widespread condensation that correlates time variables in many systems to one another. Underlying the identified variables of the condensate are additive variables of the un-condensate.

The Heisenberg Lie algebras $d\mathrm{H}(N)$ and groups have at least two natural flexings, the unitary and the orthogonal line of groups. We present them next and then choose one.

4.2   The A line

Baugh [5] regularizes the Heisenberg algebra $d\mathrm{H}(n)$ within the unitary-group Lie algebra $d\mathrm{SU}(n + 1)$. He introduces a high-dimensional linear algebra $d\mathrm{SL}(\mathrm{n} + 1; \mathbb{C})$ with generators $\Lambda^\mu{}_\nu$; $\mu = 0, 1, \ldots, n$, with extra dimension 0. The $n + 1$ generators $\Lambda^\mu{}_\mu$ (no sum) are related by $\Lambda^\mu{}_\mu = 1$ (sum!). [5]. The Baugh regularization can be written by adding an index-value 0 and setting

$$q^\mu \leftarrow \Lambda^\mu{}_0, \quad p_\mu \leftarrow \Lambda^0_\mu, r \leftarrow \Lambda^0_0. \tag{38}$$

Its regulators are all the remaining n^2 independent generators Λ^μ_ν ($\mu, \nu \neq 0$).

For a unitary representation of this abstract Lie algebra in a hermitian space with the usual positive-definite metric $\|\psi\| = \sum \psi^{\mu*}\psi^{\mu}$ it is convenient to choose the Hermitian operators

$$q^{\mu} \leftarrow \frac{1}{2}[\Lambda^{\mu}{}_0 + \Lambda^0_{\mu}], \quad p_{\mu} \leftarrow \frac{i}{2}[\Lambda^{\mu}{}_0 - \Lambda^0_{\mu}]. \tag{39}$$

4.3 The D line

Alternatively one may present the n q's and p's of $d\mathrm{H}(n)$ as singular limits of generators $o^{\mu\nu}$ of an orthogonal group $\mathrm{SO}(n+2)$:

$$q^{\mu} \leftarrow \mathsf{Q}o^{\mu\,(n+1)}, \quad p^{\mu} \leftarrow \mathsf{P}o^{\mu\,(n+2)}, r \leftarrow \mathsf{R}o^{(n+1)(n+2)}. \tag{40}$$

We have no need of odd n; and the even-n groups lie on the D line. This introduces two new index values $n+1$, $n+2$ (instead of only one for the A line) and regulators $o^{(n+1)(n+2)}$, $0^{\mu\nu}$ that must freeze out in the singular limit. It also introduces regulants Q, P, $\mathsf{R} := 1/\mathsf{N}$ that approach 0 or ∞ in the singular limit. If the singular limit has the largest possible orthogonal group $\mathrm{SO}(n)$ as symmetry group then these are the only regulants. If the limit reduces $\mathrm{SO}(n)$ to the direct sum of m smaller orthogonal groups, each of these has its own trio of regulants $\mathsf{Q}_i, \mathsf{P}_i, \mathsf{R}_i$.

Minkowski space-time in n dimensions has an orthogonal group on the D line;; Hilbert space has a unitary group on the A line. Which line shall we take? Special relativity suggests the D line of groups and quantum kinematics suggests the A line.

It would simplify matters if there were a clear hierarchic relation between special relativity and quantum theory, if one could say that quantum theory is the more fundamental, and so take the A line. But quantum theory imports its time variable from macroscopic relativity physics, and its quantum imaginary is directly linked with time by its transformation under (Wigner) time reversal. When we reverse the sign of time we reverse the sign of i.

It seems to be a useful general principle that what can transform can also change. In the present discussion this suggests that i is a variable like t; that the stable linearity of quantum theory is a real linearity, not a complex one. The decentralization of time leads us to consider the decentralization of the associated i. This happens naturally along the D line, which we tentatively follow. The choice is moot for the oscillator, because the D and A lines separate only beyond the group $\mathrm{SO}(3,1) \sim \mathrm{SL}(2,\mathbb{C})$ that we use for the general-quantized oscillator dynamics.

4.4 Flexing the dynamics

Now we general-quantize the synchronic oscillator dynamics sketched in §4.1. This requires us to express the q/c dynamics in the language of Lie algebra.

We replace the anti-Hermitian generators $\check{q} := iq$, $\check{p} := ip$ of the time-independent oscillator algebra by functions of time $\check{q}(t)$, $\check{p}(t)$ obeying the canonical equation of motion. To them and i we adjoin anti-Hermitian generators $\check{t} := it$, $\check{E} := \partial_t$, and the anti-(Hermitian) Hamiltonian $\check{H} := iH$. In the singular q/c theory $\check{t}$ commutes with q, p, i, and $\check{H}$, and $[\check{t}, \partial_t] = i$. Let us write s ∂_q, ∂_p for Fréchet derivatives with respect to the non-commuting variables $\check{q}$, $\check{p}$ [11]. We define algebra elements

$$\begin{aligned}
\dot{\check{q}} &:= \frac{1}{\hbar}[\check{H}, \check{q}], \\
\dot{\check{p}} &:= \frac{1}{\hbar}[\check{H}, \check{p}], \\
d/dt \equiv D_t &:= \partial_t + \partial_{\check{q}} \cdot \dot{\check{q}} + \partial_{\check{p}} \cdot \dot{\check{p}}
\end{aligned} \tag{41}$$

15

Then the canonical dynamical equations

$$[D_t, X] = [\partial_t, X] + \frac{1}{\hbar}[\check{H}, X]. \tag{42}$$

are required to hold for every element of the synchronic algebra.

In general $\check{H}$ is a given algebraic expression in $\check{q}, \check{p}, \check{t}$, and its commutators with them are generally not linear combinations of them. Then we must adjoin commutators iteratively until the algebra closes under commutation. For the toy oscillator, with its quadratic Hamiltonian, this step is unnecessary. The six generators $\check{q}$, $\check{p}$, i, $\check{t}$, ∂_t, $\check{H}$ already close in the q/c synchronic dynamical algebra L_D.

We drop constants and accents hereafter, writing $q, \ldots, H$ for six general-quantized anti-symmetric generators. We use the prefix anti- to remind ourselves that these are anti-Hermitian. Then the defining singular relations are

$$
\begin{array}{llllll}
[q,p] & \sim & i, & [q,i] & = & 0, & [q,t] & = & 0, & [q,\partial_t] & = & 0, & [q,H] & \sim & p, \\
 & & & [p,i]] & = & 0, & [p,t] & = & 0, & [p,\partial_t]] & = & 0, & [p,H]] & \sim & -q, \\
 & & & & & & [i,t]] & = & 0, & [i,\partial_t] & = & 0, & [i,H]] & = & 0, \\
 & & & & & & & & & [t,\partial_t] & = & -i, & [t,H] & = & 0, \\
 & & & & & & & & & & & & [\partial_t,H] & = & 0.
\end{array}
\tag{43}
$$

The canonical equations (42) are identities in virtue of these relations.

We general-quantize this algebra L_D to $d\mathrm{SO}(3,1)$, with state-vector space $u := 4\mathbb{R}$. We represent anti-time by $\hat{t} = \mathsf{T}L_{23}$, anti-energy by $E = \mathsf{E}L_{24}$, oscillator anti-position by $q \sim L_{31}$, anti-momentum by $p = \mathsf{P}L_{41}$ and the regulator by $r = \mathsf{R}L_{34}$. Both the general-quantized i and the general-quantized anti-Hamiltonian are linear in $r \sim L_{31}$, with different regulants as coefficients.

This general-quantization necessarily introduces a sixth operator, a boost $b \sim L_{12}$ that vanishes in the q/c limit. Relative to the q/c theory, the general-quantized dynamics thus requires two regulators r, b. To represent the states of the oscillator of ordinary experiments it suffices to use a representation of high casimir $L^\alpha{}_\beta L^\beta{}_\alpha$ ($\alpha, \beta = 1, 2, 3, 4, 5$) in a state-vector space U of high dimension.

To represent this dynamical q/q oscillator we may use a Clifford algebra $U = \mathrm{Cl}(Nu)$ as state-vector space.

Let us identify the six oscillator operators $\hat{q}, \hat{p}, \hat{t}, \hat{E}, \hat{r}, \hat{b}$ with multiples of the six infinitesimal generators $L_{ij} = -L_{ji} \sim x_i \partial_j - x_j \partial_i$ of $\mathrm{SO}(3,1)$. These may be represented by operators on a space u with unit vectors $\gamma_1, \ldots, \gamma_4$, and contravariant metric tensor

$$g^{-1} = -\gamma_1 \otimes \gamma_1 + \gamma_2 \otimes \gamma_2 + \gamma_3 \otimes \gamma_3 + \gamma_4 \otimes \gamma_4, \tag{44}$$

and they obey the familiar commutation relations

$$L_{ij} \times L_{kl} = \frac{1}{2}[L_{ik}g_{jl} - L_{il}g_{jk} - L_{jk}g_{il} + L_{jl}g_{ik}]. \tag{45}$$

Each of these commutation relations has the form of one of the three typical forms

$$
\begin{array}{rcl}
L_{13} \times L_{13} & = & 0, \\
L_{13} \times L_{14} & = & L_{34}, \\
L_{24} \times L_{13} & = & 0
\end{array}
\tag{46}
$$

up to a sign, according to whether the two index pairs ij and kl differ in 0, 1 or 2 indices.

Notation: We designate by γ_{ij} six second-grad basis elements of the Clifford algebra $\mathrm{Cl}(3,1)$. We write $\mathsf{L}\gamma$ for left-multiplication with γ, $\mathsf{R}\gamma$ for right-multiplication with γ, and $\mathsf{D} := \mathsf{L} - \mathsf{R}$ for commutation. All $\mathsf{L}\alpha$'s commute with all $\mathsf{R}\beta$'s.

To represent the variables of the q/q event, with their huge spectra, we introduce N anti-commuting replicas $\gamma_\mu(n)$ of the above γ_μ quartets ($\mu = 1, 2, 3, 4;\ n = 1, \ldots, N$). and set

$$\Gamma_{\mu\nu} = \frac{1}{2} \sum_n [\gamma_\mu(n), \gamma_\nu(n)]. \tag{47}$$

We use the resulting Clifford algebra

$$U := \mathrm{Cl(osc)} := \mathrm{Cl}(3N, N) \tag{48}$$

as the state-vector space of the q/q event of the general-quantized oscillator, of dimension 2^{4N} before reduction. We designate by $\widehat{L}_{ij}$ representatives of the same Lie algebra acting on $U = \mathrm{Cl}(3N, N)$, the state-vector space of the q event.

Endo U is the algebra generated by the Lγ's and the Rγ's. The physical representation of the γ_{ij} on U is

$$\gamma_{ij} \rightarrow \widehat{L}_{\mu\nu} = \mathrm{D}\Gamma_{\mu\nu}. \tag{49}$$

Let us introduce six regulants Q_{jk} as scale factors for the six $\widehat{L}_{jk}$, and represent the six variables $\widehat{q}, \widehat{p}, \widehat{t}, \widehat{E}, \widehat{b}, \widehat{r}$ as $q_{ij} = \mathsf{Q}_{ij} L_{ij}$ (no summation) according to the convention

$$
\begin{array}{llllllll}
\widehat{b} & = & -q_{12} & := & -\frac{1}{2}\mathsf{Q}_{12}\mathrm{D}L_{12}, & \widehat{q} & = & +q_{23} & := & +\frac{1}{2}\mathsf{Q}_{23}\mathrm{D}L_{23}, \\
\widehat{p} & = & +q_{24} & := & +\frac{1}{2}\mathsf{Q}_{24}\mathrm{D}L_{24}, & \widehat{t} & = & -q_{13} & := & -\frac{1}{2}\mathsf{Q}_{13}\mathrm{D}L_{13}, \\
\widehat{E} & = & -q_{14} & := & -\frac{1}{2}\mathsf{Q}_{14}\mathrm{D}L_{14}, & \widehat{r} & = & +q_{34} & := & +\frac{1}{2}\mathsf{Q}_{34}\mathrm{D}L_{34}.
\end{array} \tag{50}
$$

Then Q_{jk} is the quantum unit of the variable q^{jk}. The maximum eigenvalue of $|\widehat{r}|$ is $N\mathsf{Q}_{34}/l$.

It is sometimes helpful to designate the quantum unit of any variable v by Q_v (for "the quantum of v"). For example, $\mathsf{Q}_t = \mathsf{Q}_{13} =: \mathsf{T}$ is the quantum of time, and $\mathsf{Q}_E = \mathsf{Q}_{14}$ is the quantum of energy.

Since $q_{..}$ and $L_{..}$ are both skew-symmetric, we may take the matrix $\mathsf{Q}_{..}$ of quantum cosntants to be symmetric. A more covariant description would relate the two second-rank tensors $q_{..}$ and $L_{..}$ by a mixed fourth rank tensor $\mathsf{Q}_{..}^{..}$. The coefficients $\mathsf{Q}_{..}$ are eigenvalues of $\mathsf{Q}_{..}^{..}$.

Written out, the 15 commutation relations are

$$
\begin{array}{lllllll}
\widehat{q} \times \widehat{p} & = & +\frac{1}{2}\frac{\mathsf{Q}_q\mathsf{Q}_p}{\mathsf{Q}_r}i\widehat{r}, & \widehat{r} \times \widehat{q} & = & +\frac{1}{2}\frac{\mathsf{Q}_r\mathsf{Q}_q}{\mathsf{Q}_p}i\widehat{p}, & \widehat{p} \times \widehat{r} & = & +\frac{1}{2}\frac{\mathsf{Q}_p\mathsf{Q}_r}{\mathsf{Q}_q}i\widehat{q}, \\
\widehat{t} \times \widehat{E} & = & +\frac{1}{2}\frac{\mathsf{Q}_t\mathsf{Q}_E}{\mathsf{Q}_r}i\widehat{r}, & \widehat{r} \times \widehat{t} & = & +\frac{1}{2}\frac{\mathsf{Q}_r\mathsf{Q}_q}{\mathsf{Q}_E}i\widehat{E}, & \widehat{q} \times \widehat{t} & = & -\frac{1}{2}\frac{\mathsf{Q}_q\mathsf{Q}_t}{\mathsf{Q}_b}i\widehat{b}, \\
\widehat{E} \times \widehat{r} & = & +\frac{1}{2}\frac{\mathsf{Q}_p\mathsf{Q}_r}{\mathsf{Q}_t}i\widehat{t}, & \widehat{E} \times \widehat{q} & = & 0, & \widehat{E} \times \widehat{p} & = & +\frac{1}{2}\frac{\mathsf{Q}_E\mathsf{Q}_p}{\mathsf{Q}_b}i\widehat{b}, \\
\widehat{b} \times \widehat{E} & = & +\frac{1}{2}\frac{\mathsf{Q}_b\mathsf{Q}_E}{\mathsf{Q}_p}i\widehat{p}, & \widehat{b} \times \widehat{p} & = & -\frac{1}{2}\frac{\mathsf{Q}_b\mathsf{Q}_p}{\mathsf{Q}_E}i\widehat{E}, & \widehat{b} \times \widehat{q} & = & -\frac{1}{2}\frac{\mathsf{Q}_b\mathsf{Q}_q}{\mathsf{Q}_t}i\widehat{t}, \\
\widehat{b} \times \widehat{t} & = & +\frac{1}{2}\frac{\mathsf{Q}_E\mathsf{Q}_p}{\mathsf{Q}_q}i\widehat{q}, & \widehat{b} \times \widehat{r} & = & 0, & \widehat{p} \times \widehat{t} & = & 0.
\end{array} \tag{51}
$$

with 15 structure constants of the form $\hbar_{vw} = \mathsf{Q}_v\mathsf{Q}_w/\mathsf{Q}_u$ for $L_v \times L_w = L_u \neq 0$. These relations define the general-quantized Lie algebra $\widehat{L} = d\mathrm{SO}(3, 1)$.

Each non-zero commutation relation implies a relation between quantum constants, such as

$$
\begin{aligned}
\hbar & = \frac{\mathsf{Q}_q\mathsf{Q}_p}{\mathsf{Q}_r} \\
\hbar & = \frac{\mathsf{Q}_t\mathsf{Q}_E}{\mathsf{Q}_r}, \\
\hbar_{rq} & = \frac{\mathsf{Q}_r\mathsf{Q}_q}{\mathsf{Q}_p}, \\
& \vdots
\end{aligned} \tag{52}
$$

The Jacobi identities

$$\mathrm{D}[a \times b] = [\mathrm{D}b \times \mathrm{D}a] \tag{53}$$

relate the structure constants among themselves. For example, $\mathrm{D}t$ acting on $\widehat{q} \times \widehat{p}$ produces

$$
\begin{aligned}
\mathrm{D}\widehat{t} \cdot [\widehat{q} \times \widehat{p}] &= \mathrm{D}\widehat{t} \cdot \hbar i \widehat{r} \\
&= -\hbar^2 \widehat{E} \\
&= -\mathrm{D}[\widehat{q} \times \widehat{p}] \cdot \widehat{t} \\
&= (\mathrm{D}\widehat{q}\mathrm{D}\widehat{p} - \mathrm{D}\widehat{p}\mathrm{D}\widehat{q}) \cdot \widehat{t} \\
&= \mathrm{D}\widehat{p}\hbar_{qt} i \widehat{b}, \\
&= \hbar_{pb}\hbar_{qt} E, \\
\hbar^2 &= \hbar_{pb}\hbar_{qt}, \\
\mathsf{Q}_t \mathsf{Q}_r^2 &= \mathsf{Q}_E \mathsf{Q}_q \mathsf{Q}_p
\end{aligned}
\tag{54}
$$

To be sure, if we apply $\mathrm{D}\widehat{r}$, $\mathrm{D}b$, $\mathrm{D}p$, or $\mathrm{D}q$ to this $\widehat{q}, \widehat{p}$ commutation relation, we obtain only $0 = 0$.

The q/q history of the oscillator is a q set of q events. Its supporting vector space is therefore $U = \mathrm{Cl}\, u$.

The same relations (51) hold for the exponentially higher-dimensional Clifford algebra $V(\mathrm{M}) = \mathrm{Cl}\, T = \mathbf{2}^T$ that is the state-vector space of the dynamical history and supports the q/q dynamical algebra.

5 Consequences of general-quantization

The most salient consequence of general-quantization is that it gives all variables of the simple system finite bounded discrete spectra, without reducing the number of continuous symmetries. On the A line, the basic variables represent generators of an orthogonal group and are whole multiples of a basic quantum unit.

At the same time, the simplicity principle has virtually driven us to a revised concept of dynamics and time. A simple dynamics is not defined by a one-parameter group of unitary time translation operators. The system time is a quantum variable of discrete spectrum and its advance is not a unitary transformation. Near the beginning and ending of the system time, at $|t| \sim \pm \mathsf{N}\mathsf{T}$, the multiplicities of the eigenvalues of $|t|$ vary so rapidly with t — namely linearly in the difference $|t| - \max|t|$ — that it is a bad approximation to suppose that the different values of $|t|$ have isomorphic eigenspaces related by unitary transformations. In middle times, $|t| \ll \max|t|$, unitarity is a good approximation. In this toy model, space becomes small as time nears its beginning or end. A similar thing happens near the beginning of time in general relativity too, but it is too soon to say whether the two phenomena are related.

The relation between the quantities i (a constant) and $\widehat{H}$ (a variable) of the old q/c synchronic dynamics now resembles that between constant mass and variable energy of the old Newtonian physics. $\widehat{H}$ and i are actually the same variable r seen through different lenses. The window through which we see i covers the entire range of values of $\widehat{r}$, and in ordinary situations $\widehat{r}$ remains so close to its extreme value that it can be treated as constant. That constant, rescaled to unit magnitude, is i. On the other hand $\widehat{r}$ is not exactly constant, and its departure from its extreme value, again suitably rescaled, is $\widehat{H}$.

Each previous general-quantization has introduced new forms of energy with important consequences. It remains to be seen what consequences this new concept of energy will have.

Since quantum theory began as a regularization procedure of Planck, it is rather widely accepted that further regularization of present quantum physics calls for further quantization, but what to

quantize and how to quantize it remains at least a bit unclear. Now we regard quantization as a special case of flexing, and a path becomes clearer. It is marked by singular Killing forms ripe for flexing. All the singular groups and infinities of present physics arguably result from flattening, and this originates in a preference for singular groups over regular ones that is not based on experiment but on ideology.

Flexing the time-independent linear harmonic oscillator results in a finite quantum theory with three quantum constants $\hbar, \hbar', \hbar''$ instead of the usual one. This finite quantum oscillator is isomorphic to a dipole rotator with $N = 2l + 1 \sim 1/(\hbar'\hbar'') \gg 1$ states and bounded Hamiltonian $H = A(L_1)^2 + B(L_2)^2$. Its position and momentum variables are quantized with uniformly spaced bounded finite spectra and supposedly universal quanta of position and momentum. For fixed quantum constants and large $N \gg 1$ there are three broad classes of finite oscillator, soft, medium, and hard. The field oscillators responsible for infra-red and ultraviolet divergences are soft and hard respectively. Medium oscillators have $\sim \sqrt{N}$ low-lying states having nearly the same zero-point energy and level spacing as the quantum oscillator and nearly obeying the Heisenberg uncertainty principle and the equipartition principle. The corresponding rotators are nearly polarized along the z axis with $L_3 \sim \pm l$.

The soft and hard oscillators have infinitesimal 0-point energy, and grossly violate both equipartition and the Heisenberg uncertainty relation. They do not resemble the quantum oscillator at all. Their low-lying energy states correspond to rotators with $L_1 \sim 0$ or $L_2 \sim 0$ instead of $L_3 \sim \pm l$. Soft oscillators have frozen momentum $p \approx 0$ because their maximum potential energy is too small to produce one quantum of momentum. Hard oscillators have frozen position $q \approx 0$ because their maximum kinetic energy is too small to produce one quantum of position.

The zero-point energy of a physical oscillator likely contributes to its gravitational field. It will be interesting to estimate its contribution to astronomical gravitational fields. For a consistent estimate we should regularize the space-time operators x^μ, ∂_μ as well as the canonical field variables q, p, since both algebras have the same instability. This changes not only the structure of the individual oscillators, as considered here, but also the number and distribution of the oscillators. We leave this study for later.

The finite quantum theory modifies low- and high-energy physics. Because the low-lying energy levels of medium oscillators have nearly uniform spacing, the energy of two excitations is but slightly less than the sum of their separate energies. The corresponding quanta nearly do not interact, and the small interaction that they have is attractive. For soft or hard oscillators, the energy level varies quadratically with the energy quantum number. The energy of two quanta of oscillation is twice the sum of their separate energies, for example. The corresponding quanta have a repulsive interaction of great strength; the interaction energy is equal to the total energy of the separate quanta. Thus the simplest regularization leads to interactions between the previously uncoupled excitation quanta of the oscillator, weakly attractive for medium quanta, strongly repulsive for soft or hard quanta.

Like Dirac's theory of the "anomalous" magnetic moment of the relativistic electron, these extreme-energy effects depend on factor ordering. They can be adjusted to fit the data by re-ordering factors and so are not crucial tests of the theory. The theory of a more physical system will be necessary for that.

6   Acknowledgments and references

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